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CONDUCTION WITH TIME-DEPENDENT HEAT SOURCES AND BOUNDARY CONDITIONS

(A MODIFIED SEPARATION-OF-VARIABLES TECHNIQUE)

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Abstract—A procedure for extending the separation-of-variables technique to heat-conduction problems with time-dependent heat sources and boundary conditions is presented. It is shown how a modification of the "separation" method, similar to one made when seeking vibration solutions and applicable to many linear partial-differential equations, transforms problems of this general class into a set of transient and steady-state sub-problems for which solution methods are well established. Results take the form of quasi-static expressions superimposed upon a product series involving characteristic functions of the corresponding uniformly excited cases. The method is thus suitable for extending existing solutions to time-dependent heat-source and end conditions, as well as offering an alternative and occasionally more convenient approach from better-known integral methods.

NOMENCLATURE

$F(P)$, $h(x)$,	arbitrary, but integrable, space variables;
M,	film-coefficient parameter;
N,	conductivity parameter;
$Q(P, t)$, $Q_j(P)$,	heat-generation rates per unit volume;
$T(P, t)$,	temperature;
$T_{0i}(P)$, $T_{0j}(P)$,	static temperature functions;
V_P,	volume in P-space;
a_t,	thermal diffusivity;
b_n, c_{in}, d_n,	coefficients of series expansions;
$f_i(t)$, $g_j(t)$,	arbitrary, but integrable, time functions;
k,	thermal conductivities;
t,	time variable;
α_n^2, λ_n,	eigenvalues and separation constants;
$X_n(x)$, $\varphi_n(P)$,	eigenvectors;
ν,	outward drawn normal on boundary, B;
$\psi_n(t)$,	separation function (of t);
$(\dot{\ })$,	$\partial(\ )/\partial t$;
$(\ )'$, $(\ )''$,	$\partial(\ )/\partial x$, $\partial^2(\ )/\partial x^2$.

INTRODUCTION

HEAT-CONDUCTION problems in which boundary temperatures, heat fluxes, or internal energy sources are prescribed functions of time are commonly encountered in practice. Although solution techniques for certain cases, characterized by unsteady "inputs", exist, the classical method of separating variables is not directly applicable.

This paper adapts an existing modification of the "separation" approach for treating problems of the types enumerated. Although sufficiently broad and easy to use in principle, the method presented does not appear in the standard repertory of conduction solution methods [1–4]. However, essentially similar approaches have been employed to solve time-dependent boundary condition, continuum vibration problems [5–9].

METHOD

Consider the equation:

$$\nabla^2 T(P, t) = a_t^{-1}\, \partial T(P, t)/\partial t, \quad \text{for } P \text{ in } D,\ t > 0 \tag{1}$$

with

$$M\, \partial T/\partial \nu + N\, T = \sum_i f_i(t)\, G_i(P), \quad \text{for } P \text{ on } B,\ t > 0 \tag{2}$$

and

$$T = F(P), \quad \text{for} \quad P \text{ in } D,\ t = 0. \tag{3}$$

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D is a continuum domain with boundaries at $P = B$.

For solutions of (1–3), assume

$$T(P, t) = \sum_n \varphi_n(P)\,\psi_n(t) + \sum_i T_{oi}(P) f_i(t) \quad (4)$$

in which the $T_{oi}(P)$ are harmonic, i.e.

$$\nabla^2 T_{oi} = 0. \quad (5)$$

Letting

$$T_{oi} = \sum_n c_{in}\,\varphi_n, \quad (6)$$

substituting (4) into (1), and (6) into the result yields

$$\sum_n \nabla^2\varphi_n(P)\,\psi_n(t) + \sum_i \nabla^2 T_{oi} f(t) = 1/a_t[\sum_n \varphi_n(P)\,\dot{\psi}_n(t) + \sum_i \dot{f}_i(\sum_n c_{in}\,\varphi_n)].$$

Making use of (5), equating termwise, dividing by $\varphi_n\,\psi_n$, and choosing $-\lambda_n$'s for separation constants gives

$$\frac{\nabla^2\varphi_n}{\varphi_n} = 1/a_t \frac{\dot{\psi}_n + \sum_i c_{in}\,\dot{f}_i(t)}{\psi_n} = -\lambda_n. \quad (7)$$

Substituting (4) into the boundary conditions (2) gives

$$\sum_n \psi_n(t)\,[M\,\partial\varphi_n(B)/\partial\nu + N\,\varphi_n(B)] = \sum_i f_i(t)\,\{G_i(B) - [M\,\partial T_{oi}(B)/\partial\nu + N\,T_{oi}(B)]\}. \quad (8)$$

The following boundary conditions are deduced from (8):

$$M\,\partial\varphi_n/\partial\nu + N\,\varphi_n = 0, \text{ on } B \quad (9)$$

$$M\,\partial T_{oi}/\partial\nu + N\,T_{oi} = G_i, \text{ on } B. \quad (10)$$

Substitution of (4) into the initial condition (3) yields

$$\sum_n \varphi_n\,\psi_n(0) + \sum_i T_{oi}\, f_i\,(0) = F(P)$$

which, upon insertion of (6) and

$$T(P, 0) = F(P) \equiv \sum_n b_n\,\varphi_n(P), \quad (11)$$

becomes

$$\sum_n \varphi_n\,[\psi_n(0) + \sum_i c_{in}\, f_i(0) - b_n] = 0.$$

Thus we take

$$\psi_n = b_n - \sum_i c_{in}\, f_i, \quad \text{for } t = 0. \quad (12)$$

(1), (2) and (3) have now been transformed into the following three standard-type sub-problems:

(a) From (7) and re-writing (9):

$$\nabla^2\varphi_n + \lambda_n\,\varphi_n = 0 \text{ in } D \text{ and}$$
$$M\,\partial\varphi_n/\partial\nu + N\,\varphi_n = 0 \text{ on } B. \quad (7' \text{ and } 9)$$

(b) From (5) and (10):

$$\nabla^2 T_{oi} = 0 \text{ in } D \text{ and}$$
$$M\,\partial T_{oi}/\partial\nu + N\,T_{oi} = G_i \text{ on } B. \quad (5 \text{ and } 10)$$

(c) From (7) and rewriting (12):

$$\dot{\psi}_n + \lambda_n\,a_t\,\psi_n = -\sum_i c_{in}\dot{f}_i(t), \quad t > 0 \quad (7'')$$

$$\psi_n = b_n - \sum_i c_{in} f_i(0), \qquad t = 0. \quad (12)$$

To evaluate the b_n's and c_{in}'s, use is made of the orthogonality condition

$$\int_D \varphi_m\varphi_n\,\mathrm{d}V_P = 0, \qquad m \neq n. \quad (13)$$

Thus,

$$b_n = \frac{\int_D F(P)\,\varphi_n(P)\,\mathrm{d}V_P}{\int_D \varphi_n^2\,\mathrm{d}V_P}, \quad (14)$$

$$c_{in} = \frac{\int_D T_{oi}(P)\,\varphi_n(P)\,\mathrm{d}V_P}{\int_D \varphi_n^2\,\mathrm{d}V_P}. \quad (15)$$

For the case of time-dependent heat sources and homogeneous boundary and initial conditions the governing equations and conditions are

$$\nabla^2 T(P, t) + k^{-1}\sum_j Q_j(P)\,g(t) = a_t^{-1}\,\partial T(P, t)/\partial t, \quad t > 0, \quad (16)$$

$$M\,\partial T/\partial\nu + N\,T = 0 \text{ on } B \text{ with } t > 0, \quad (17)$$

$$T = 0 \text{ in } D \text{ when } t = 0. \quad (18)$$

To solve, assume a solution of the form

$$T(P, t) = \sum_n \varphi_n(P)\,\psi_n(t) + \sum_j T_{oj}(P)\,g(t). \quad (19)$$

Letting

$$T_{oj}(P) = \sum_n \mathrm{d}_{jn}\,\varphi_n(P) \text{ and } \nabla^2 T_{oj} + k^{-1}\,Q_j = 0, \quad (20)$$

it can be shown that (16–18) reduce to the following sub-problems:

(a) $\nabla^2\varphi_n + \lambda_n \varphi_n = 0$ in D with

$$M\, \partial\varphi_n/\partial\nu + N\varphi_n = 0 \text{ on } B \quad (21)$$

(b) $\nabla^2 T_{oj} + k^{-1} Q_j = 0$ in D and

$$M\, \partial T_{oj}/\partial\nu + N\, T_{oj} = 0 \text{ on } B \quad (22)$$

(c) $\dot\psi_n + a_t\, \lambda_n\, \psi_n = -\sum_j d_{jn}\, \dot g_j,\ t > 0$ (23)

$$\psi_n = -\sum_j d_{jn}\, g_j, \qquad t = 0. \quad (24)$$

An example

Given: the one-dimensional transient heat equation:

$$\partial^2 T/\partial x^2 = a_t^{-1}\, \partial T/\partial t \quad (25)$$

with boundary conditions:

$$T(0, t) = f(t), \qquad T(L, t) = 0 \quad (26)$$

and initial condition:

$$T(x, 0) = h(x) \quad (27)$$

To solve (25–27), let

$$T(x, t) = \sum_n X_n(x)\, \psi_n(t) + T_0(x) f(t) \quad (28)$$

in which T_0 satisfies the steady-state heat-conduction equation

$$\mathrm{d}^2 T_0/\mathrm{d}x^2 = 0 \quad (29)$$

with boundary conditions to be prescribed shortly.

Substituting (28) into (25–27) one obtains the transformed equation

$$\sum_n X_n''\, \psi_n + T_0''\, f(t) = a_t^{-1} \sum_n X_n\, \dot\psi_n + a_t^{-1}\, T_0\, \dot f \quad (30)$$

boundary conditions

$$\sum_n X_n(0)\, \psi_n(t) = f\, [1 - T_0(0)] \quad (31a)$$

$$\sum_n X_n(L)\, \psi_n(t) = -f\, T_0(L), \quad (31b)$$

and initial condition

$$\sum_n X_n(x)\, \psi_n(0) + T_0\, f(0) = h(x). \quad (32)$$

Expanding T_0 and h in series of X_n functions (which are orthogonal and assumed to constitute a complete set),

$$T_0(x) = \sum_n c_n\, X_n(x) \quad (33a)$$

$$h(x) = \sum_n d_n\, X_n(x), \quad (33b)$$

substituting (33a) into (30), making use of (29), equating termwise in n and dividing through by $X_n\, \psi_n$, transforms (30) into

$$X_n''/X_n = a_t^{-1}\, (\dot\psi_n + c_n\, \dot f)/\psi_n \equiv -\alpha_n^2$$

in which $-\alpha_n^2$ is chosen as separation constant. Therefore,

$$X_n'' + \alpha_n^2\, X_n = 0 \quad (34)$$

and

$$\dot\psi_n + \alpha_n^2\, a_t\, \psi_n = -c_n\, \dot f. \quad (35)$$

Boundary and initial conditions for (34) and (35) are obtained after (31a), (31b) and (32) are examined, i.e.

$$X_n(0) = X_n(L) = 0 \quad (36a)$$

$$\psi_n(0) = d_n - c_n f(0) \quad (36b)$$

and

$$T_0(0) = 1, \qquad T_0(L) = 0. \quad (36c)$$

Solution of the pairs {(29), (36c)}, {(34), (36a)} and {(35), (36b)} is straightforward after the orthogonality of the X_n's is used to evaluate the c_n's and d_n's:

$$c_n = \frac{\int_0^L T_0(x)\, X_n(x)\, \mathrm{d}x}{\int_0^L X_n^2\, \mathrm{d}x}$$

$$d_n = \frac{\int_0^L h(x)\, X_n(x)\, \mathrm{d}x}{\int_0^L X_n^2\, \mathrm{d}x}.$$

It should be noted that a solution is also possible if, instead of having T_0 satisfy (29), it is taken to be an arbitrary function of x which also satisfies (36c) and possess continuous first and second derivatives. For this case, (35) would read:

$$\dot\psi_n + \alpha_n^2\, a_t\, \psi_n = e_n\, f - c_n\, \dot f \quad (37)$$

with

$$T_0''(x) = \sum_n e_n\, X_n(x) \quad (38)$$

and

$$e_n = \int_0^L T_0''\, X_n\, \mathrm{d}x \Big/ \int_0^L X_n^2\, \mathrm{d}x.$$

This approach corresponds to Mindlin and Goodman's method [6] for beam-vibration problems. Timoshenko [7] and Archer [9] modified this by superimposing a quasi-static solution upon the free-vibration modes. This latter technique, which is comparable to the one used here, has the advantage of eliminating at least one forcing term in the separated time equation [e.g. $f(t)$ in (37)] and requires at least one less eigenvector expansion [e.g. (38) is unnecessary]. Furthermore, it is easier to physically interpret the significance of superposing a quasi-static component upon an eigenvalue-product solution. One may also expect that a quicker series convergence in (28) is effected when the time is large and the "transient" component of the solution is small with respect to the "steady-state" solution component.

DISCUSSION

Superposition of the two general cases treated leads to a systematic sub-division of the original problem into a series of quasi-static ones superimposed upon a simplified transient system. These latter components are composed of eigenvectors and corresponding time terms. The time terms in the solutions to the transient problems satisfy first-order differential equations in which the forcing functions are governed by the time-dependent boundary and heat-source effects.

Implicit in the derivations is the assumption that the time-dependent excitations can be put in terms of sums of separable space and time functions. This presents no serious limitation upon the technique, since a large majority of actual situations satisfy this condition. Furthermore, the method assumes that an eigenvalue solution may be associated with the one-, two- or three-dimensional domain considered. This requires that the body be finite in at least one dimension.

The present technique is reminiscent of the solution of ordinary linear nonhomogeneous equations in which general and particular solutions are sought. The technique may thus be thought of as a quasi-steady solution superimposed upon a complementary (or "transient") one. The former account for external disturbances while the latter are composed of elements which are intrinsic to the system (i.e. eigenvalues).

The present method, besides supplying a new approach to the type of problem and boundary conditions discussed, takes advantage of the fact that existing steady boundary and heat-source eigenvalue solutions can be converted into corresponding time-varying "forced" solutions. Furthermore, since it is not always possible or convenient to perform the integration which appears in existing methods such as Duhamel's integral theorem or Green's functions, nor to invert a relevant Laplace transform, the procedure outlined here may supply solutions to problems for which existing methods are deficient.

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Résumé—L'auteur présente une extension de la méthode de séparation des variables aux problèmes de conduction thermique avec sources de chaleur et conditions aux limites fonctions du temps. Il montre comment une modification de la méthode de "séparation" semblable à celle utilisée dans la méthode des solutions vibratoires et applicable à plusieurs équations aux dérivées partielles linéaires transforme les problèmes de ce type général en un ensemble de problèmes relatifs aux phénomènes stationnaires et transitoires dont les méthodes de résolution sont parfaitement établies. Les résultats ont la forme d'expressions quasi-stationnaires superposées à des séries de produits contenant les fonctions caractéristiques correspondant aux cas où l'excitation estuniforme. La méthode permet donc d'étendre les solutions existantes aux cas des sources de chaleur avec conditions limites fonctions du temps et permet en outre de trouver une solution approchée nouvelle, qui est quelquefois plus commode, à partir des méthodes intégrales bien connues.

Zusammenfassung—Nach dem angegebenen Verfahren lässt sich die Methode der Trennung der Variablen auf Wärmeleitprobleme mit zeitabhängigen Wärmequellen und Grenzbedingungen ausdehnen. Es wird gezeigt, wie eine Modifikation der "Trennmethode", wie sie bei Schwingungsproblemen und vielen partiellen Differentialgleichungen angewandt wird, vom allgemeinen Problem auf eine Reihe von Unterproblemen für stationäre Verhältnisse führt, für die Lösungen bekannt sind. Die Ergebnisse zeigen Ausdrücke quasistationärer Vorgänge, die einer Produktenreihe aus charakteristischen Funktionen der entsprechenden Fälle überlagert sind. Die Methode eignet sich sowohl dafür, bestehende Lösungen auf zeitabhängige Wärmeerzeugung und Grenzbedingungen auszudehnen als auch eine weitere, gelegentlich bequemere Annäherung nach bereits bekannten Integralmethoden zu erreichen.

Аннотация—Даётся способ применения метода разделения переменных к задачам теплопроводности при наличии зависящих от времени источников тепла и граничных условий. Показано, как модификация метода «разделения», аналогичная модификации, проведенной при поиске колеблющихся решений и применимой ко многим линейным дифференциальным уравнениям в частных производных, преобразует задачи этого общего класса в систему вспомогательных задач переходного и стационарного состояний, для которых хорошо разработаны методы решения. В результате получаются квазистатические условия, наложенные на ряды произведений, которые включают характеристические функции соответствующих слчаев равномерного возбуждения. Таким образом, этот способ позволяет применить существующие решения к случаю наличия зависящего от времени источника тепла и конечных условий, а также выбрать более подходящий и иногда бодее удобный метод из известных интегральных методов.